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Spectral finite element of a helix

H. Shahsavari *, M. Ostoja-Starzewski

Department of Mechanical Engineering, McGill University, 817 Sherbrooke Street West, Montréal, Québec, Canada H3A 2K6

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Abstract

Determination of a spectral (i.e. frequency dependent) finite element of a helix is the focus of this communication. The helix is treated as straight, linear elastic element, exhibiting coupling of axial with torsional responses. We derive explicit forms of all the coefficients of the stiffness matrix and plot their dependencies on the frequency and the parameter describing the said coupling. In general, the growth of that parameter leads to a progressively denser occurrence of the resonances of both axial and torsional motions.

Keywords: Helix; Wire rope; Wave propagation; Spectral finite element; Vibration

1. Background

To a mechanician, a helix is a structural element coupling axial and torsional responses. This comes about through some kind of a helically-wound microstructure, such as in the very well known wire ropes, Fig. 1. The book by Costello (1997) discusses the latter subject extensively, albeit in the static setting. The said coupling is expressed by two constitutive equations

$$F = A_1\varepsilon + A_2\tau, \quad M = A_3\varepsilon + A_4\tau, \tag{1}$$

where F is tensile force and M is torque, while ε is axial strain and τ is angle of twist per unit length. Furthermore, A_1, A_2, A_3, A_4 are constitutive constants dependent on both the rope material and construction. These constants are necessarily positive, and also these relations hold

$$A_2 = A_3, \quad A_1A_4 \pm A_2A_3 > 0. \tag{2}$$

Notably, there are various other examples of helices in engineering and in nature, but they are all characterized by the same equation system as (1). The equations of motion of a helix, referred to the unstressed rope length and in the absence of body forces, become

* Corresponding author. Fax: +1-514-398-7365.
E-mail address: martin.ostoja@mcgill.ca (H. Shahsavari).

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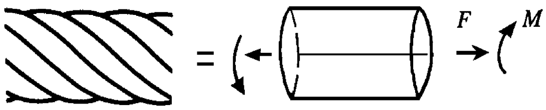

Fig. 1. A wire rope structure (left) shows the coupling of axial and torsional responses, typical of a helix.

$$\frac{\partial F}{\partial x} = m\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial M}{\partial x} = J\frac{\partial^2 \theta}{\partial t^2}. \tag{3}$$

Here m and J are, respectively, mass and mass moment of inertia per unit unstressed length. Substituting Eq. (1) into (3), the equations governing coupled extensional–torsional oscillations of wire rope become

$$A_1\frac{\partial^2 u}{\partial x^2} + A_2\frac{\partial^2 \theta}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2}, \quad A_3\frac{\partial^2 u}{\partial x^2} + A_2\frac{\partial^2 \theta}{\partial x^2} = J\frac{\partial^2 \theta}{\partial t^2}. \tag{4}$$

2. Spectral stiffness matrix

Let us now consider harmonic motions according to:

$$u(z,t) = \hat{u}(x,\omega)e^{i\omega t}, \quad \theta(z,t) = \hat{\theta}(x,\omega)e^{i\omega t}, \tag{5}$$

where ω is the frequency, and the hat stands for a quantity in the frequency space (Doyle, 1991; Ostoja–Starzewski and Woods, 2003). Now, of interest is the derivation of a spectral stiffness matrix expressing a connection between the kinematic and dynamic quantities at both ends of a helical element:

$$\begin{Bmatrix} \widehat{F}_1 \\ \widehat{F}_2 \\ \widehat{M}_1 \\ \widehat{M}_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{Bmatrix}. \tag{6}$$

Here the subscripts 1 and 2 of F and M denote the left and right ends of the rope, respectively.

The derivation of the spectral stiffness matrix follows these steps. First, substituting Eq. (5) into Eq. (4) one obtains:

$$A_1\frac{\partial^2 \hat{u}}{\partial x^2} + A_2\frac{\partial^2 \hat{\theta}}{\partial x^2} = -m\omega^2\hat{u}, \quad A_3\frac{\partial^2 \hat{u}}{\partial x^2} + A_4\frac{\partial^2 \hat{\theta}}{\partial x^2} = -J\omega^2\hat{\theta}. \tag{7}$$

To solve Eq. (7) we consider:

$$\hat{u}(x,\omega) = ue^{ikz}, \quad \hat{\theta}(x,\omega) = \theta e^{ikz}. \tag{8}$$

Here, u and θ are constant wave amplitudes, and k is the wave number. As shown in (Samras et al., 1974), there are two wave speeds c_1 and c_2

$$c_{1,2}^2 = \frac{2(A_1A_4 - A_2A_3)}{(A_1J + A_4m) \pm \left[(A_1J - A_4m)^2 + 4mJA_2A_3\right]^{1/2}}, \tag{9a}$$

while the ratios R_1 and R_2 of torsional to extensional oscillations' amplitudes are

$$R_{1,2} = \frac{\theta}{u} = \frac{(mc_{1,2}^2 - A_1)}{A_2}. \tag{9b}$$

with this, the solution for extensional–torsional oscillations of wire rope in terms of trigonometric functions becomes

$$\hat{u}(x,\omega) = U_1 \sin(k_1 x) + U_2 \sin(k_1(L-x)) + U_3 \sin(k_2 x) + U_4 \sin(k_2(L-x)) \tag{10a}$$

$$\hat{\theta}(x,\omega) = R_1[U_1 \sin(k_1 x) + U_2 \sin(k_1(L-x))] + R_2[U_3 \sin(k_2 x) + U_4 \sin(k_2(L-x))]. \tag{10b}$$

In the above, U_1 through U_4 are arbitrary constants. Thus, in effect, there are two waves: a predominantly extensional wave traveling at a speed c_1, and a predominantly torsional wave traveling at a speed c_2. Next, by taking the boundary conditions as

$$u(0) = \hat{u}_1 \quad u(L) = \hat{u}_2 \quad \theta(0) = \hat{\theta}_1 \quad \theta(L) = \hat{\theta}_2, \tag{11}$$

and substituting (11) into (10) we find

$$\begin{bmatrix} 0 & \sin(k_1L) & 0 & \sin(k_2L) \\ \sin(k_1L) & 0 & \sin(k_2L) & 0 \\ 0 & R_1\sin(k_1L) & 0 & R_2\sin(k_2L) \\ R_1\sin(k_1L) & 0 & R_2\sin(k_2L) & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix}, \tag{12}$$

with the constants in terms of nodal deformations being

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \frac{1}{(R_2 - R_1)} \begin{bmatrix} (\hat{u}_2 R_2 - \hat{\theta}_2)\csc(k_1L) \\ (\hat{u}_1 R_2 - \hat{\theta}_1)\csc(k_1L) \\ -(\hat{u}_2 R_1 - \hat{\theta}_2)\csc(k_2L) \\ -(\hat{u}_1 R_1 - \hat{\theta}_1)\csc(k_2L) \end{bmatrix}. \tag{13}$$

Now, differentiating Eq. (10) with respect to x, and using Eqs. (1) and (13), we find

$$\begin{aligned} \widehat{F}_1 = \frac{-1}{(R_2 - R_1)} \Big[& \{-k_1R_2E_1\cot(k_1L) + k_2R_1E_2\cot(k_2L)\}\hat{u}_1 + \{k_1R_2E_1\csc(k_1L) - k_2R_1E_2\csc(k_2L)\}\hat{u}_2 \\ & + \{k_1E_1\cot(k_1L) - k_2E_2\cot(k_2L)\}\hat{\theta}_1 + \{-k_1E_1\csc(k_1L) + k_2E_2\csc(k_2L)\}\hat{\theta}_2 \Big] \end{aligned} \tag{14}$$

where we replaced U_1 through U_4 according to (13) and introduced

$$E_1 = A_1 + A_2R_1, \quad E_2 = A_1 + A_2R_2, \quad E_3 = A_2 + A_4R_1, \quad E_4 = A_2 + A_4R_2. \tag{15}$$

Proceeding in the same fashion for $\widehat{F}_2$, $\widehat{M}_1$, and $\widehat{M}_2$, we determine the four-by-four spectral stiffness matrix of (6) as follows:

$$K = \frac{1}{R_2 - R_1} \begin{bmatrix} k_1R_2E_1\cot(k_1L) - k_2R_1E_2\cot(k_2L) & -k_1R_2E_3\csc(k_1L) + k_2R_1E_4\csc(k_2L) & -k_1E_1\cot(k_1L) + k_2E_2\cot(k_2L) & k_1E_1\csc(k_1L) - k_2E_2\csc(k_2L) \\ -k_1R_2E_1\csc(k_1L) + k_2R_1E_2\csc(k_2L) & k_1R_2E_1\cot(k_1L) - k_2R_1E_2\cot(k_2L) & k_1E_1\csc(k_1L) - k_2E_2\csc(k_2L) & -k_1E_1\cot(k_1L) + k_2E_2\cot(k_2L) \\ k_1R_2E_3\cot(k_1L) - k_2R_1E_4\cot(k_2L) & -k_1R_2E_3\csc(k_1L) + k_2R_1E_4\csc(k_2L) & -k_1E_3\cot(k_1L) + k_2E_4\cot(k_2L) & -k_1R_2E_3\csc(k_1L) + k_2R_1E_4\csc(k_2L) \\ -k_1R_2E_3\csc(k_1L) + k_2R_1E_4\csc(k_2L) & k_1R_2E_3\cot(k_1L) - k_2R_1E_4\cot(k_2L) & k_1E_3\csc(k_1L) - k_2E_4\csc(k_2L) & -k_1E_3\cot(k_1L) + k_2E_4\cot(k_2L) \end{bmatrix}. \tag{16}$$

Symmetry of the stiffness matrix requires that, for example $k_{41} = k_{14}$, and so, by comparing these components, we note these implications

$$R_2E_3 = -E_1 \Rightarrow R_2(A_3 + A_4R_1) = -(A_1 + A_2R_1) \tag{17a}$$

and

$$R_1E_4 = -E_2 \Rightarrow R_1(A_2 + A_4R_2) = -(A_1 + A_2R_2). \tag{17b}$$

One can easily verify that Eqs. (17a) and (17b) hold for arbitrary values of A_1 through A_4.

3. Conclusions

- Given the above formulas, in Fig. 2 we now plot all four coefficients k_{11}, k_{12}, k_{13}, k_{14} in function of the helical coupling parameter $A_2 = A_3$ at one fixed frequency: $\omega = 40$ kHz. Note that these coefficients cover all the distinct dependencies of the spectral stiffness matrix on ω. Evident here is the progressively denser location of 'hills' and 'valleys'—indicative of the occurrence of resonance—with the coupling increasing. To display the entire dependence of k_{11} through k_{14} on $A_2 = A_3$ and ω, we next plot Figs. 3–6. Overall, these k_{ij}'s appear as irregular wavy surfaces, but, for any fixed frequency, there is the same trend as already shown in Fig. 2.
- Taking $A_2 = A_3 = 0$ (absence of the coupling effect) and partitioning the four-by-four spectral stiffness matrix into four two-by-two sub-matrices as

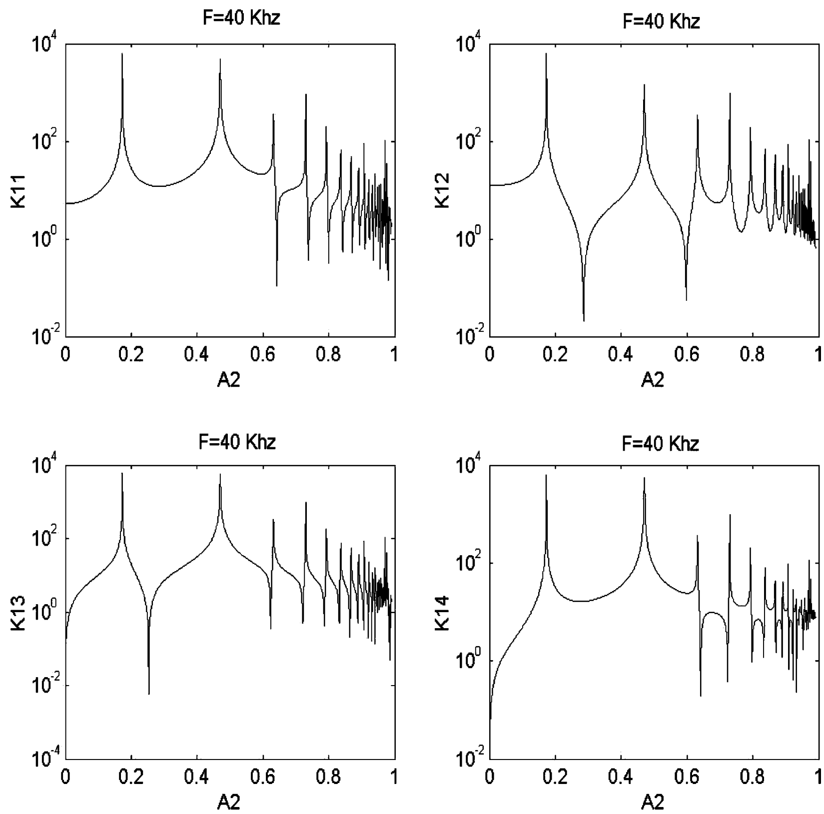

Fig. 2. Dependence of spectral stiffnesses k_{11}, k_{12}, k_{13}, and k_{14} on the axial–torsional coupling.

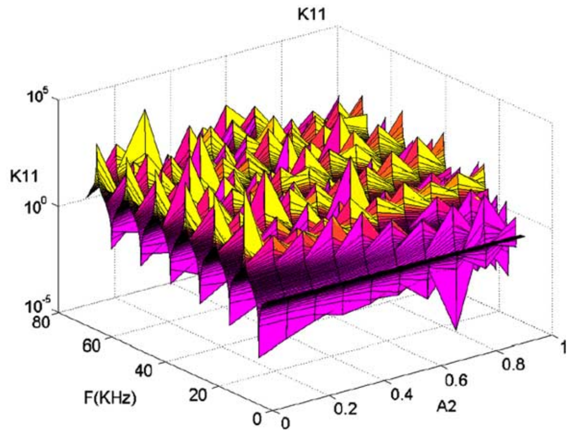

Fig. 3. Dependence of the spectral stiffness k_{11} on frequency and axial–torsional coupling.

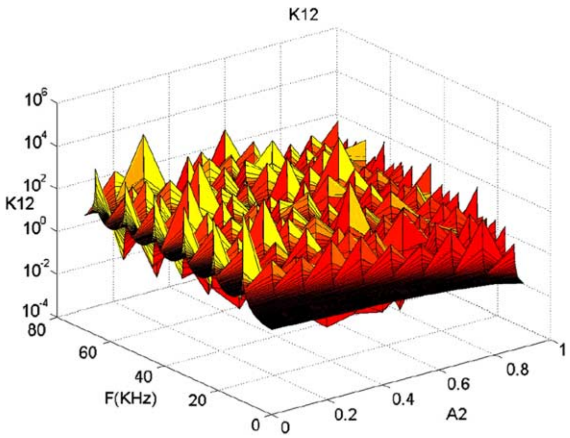

Fig. 4. Dependence of the spectral stiffness k_{12} on frequency and axial–torsional coupling.

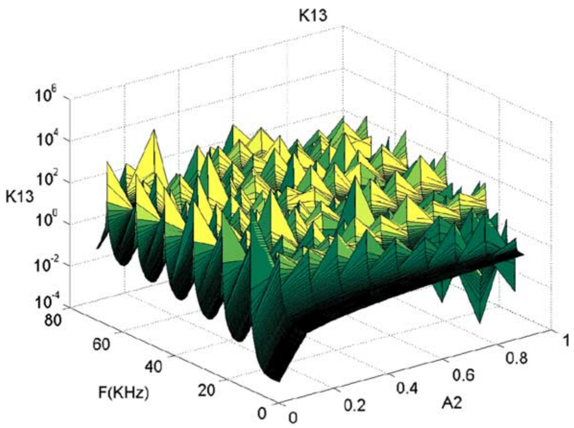

Fig. 5. Dependence of the spectral stiffness k_{13} on frequency and axial–torsional coupling.

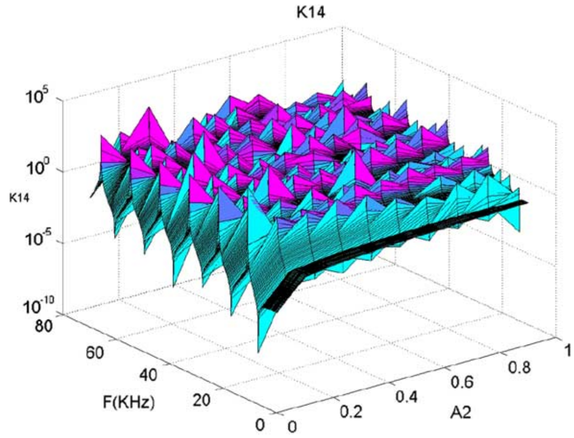

Fig. 6. Dependence of the spectral stiffness k_{14} on frequency and axial–torsional coupling.

$$K = \begin{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} & \begin{bmatrix} k_{13} & k_{14} \\ k_{23} & k_{24} \end{bmatrix} \\ \begin{bmatrix} k_{31} & k_{32} \\ k_{41} & k_{42} \end{bmatrix} & \begin{bmatrix} k_{33} & k_{34} \\ k_{43} & k_{44} \end{bmatrix} \end{bmatrix}, \qquad (18)$$

we see that the first and second sub-matrices on the diagonal represent the spectral stiffness matrices of a straight bar under a pure axial and torsional load, respectively, with the two off-diagonal sub-matrices being zero identically. As $A_2 = A_3$ increase from zero up, two sub-matrices on the diagonal get modified and two off-diagonal sub-matrices (equal to each other due to the already mentioned symmetry) come into existence.

Related recent work has focused on harmonic waves in thermoelastic helices with either parabolic or hyperbolic heat conduction (Ostoja-Starzewski, 2003). In general, both mechanical waves (axial and torsional) are speeded up and damped as the thermoelastic coupling constant grows.

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