Response of a Helix Made of a Fractional Viscoelastic Material

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Under investigation is the effective response of a helical strand (helix) made of a viscoelastic material governed by a constitutive relation with fractional-order (i.e., not integer-order) derivatives. The relation involves a 5-parameter model, which is well known to represent a real response much better than the conventional, integer-order models with the same number of parameters. We employ the correspondence principle of viscoelasticity to pass from the level of the strand's material to that of an effective, coupled axial-torsional response of the helix. The resulting fractional-order differential equation is more complex (i.e., it involves higher derivatives) than the constitutive equation governing the material per se. Also, the use of a fractional-order model results in more complexity of the helix' effective viscoelastic response than does an integer-order model with the same number of parameters. It is shown that shear deformations are more important than dilatational deformations. Lastly, a standard relaxation test is studied and an analytic solution is derived. [DOI: 10.1115/1.2745401]

1 Introduction

The term helix means a single helical strand or a bundle of such strands. In the latter case, such as a wire rope, there is a straight strand at the core, surrounded by several outer helical strands, Fig. 1. In the case of a bundle without the core strand, the outer helical strands are assumed to not collapse, and to not interact on their contact surfaces. Effectively, the loads they carry add up as in a parallel system.

The effective constitutive equations of the helix involve coupling of axial with torsional responses (e.g. [1])

$$\sigma = C_1\varepsilon + C_2\tau$$

$$\mu = C_3\varepsilon + C_4\tau \tag{1}$$

Here σ is the axial stress and μ is the couple-stress (moment per unit area), while ε and τ are the axial strain and angle of twist per unit length respectively. $C_1 \cdots C_4$ are the constitutive coefficients dependent on the material properties and the 3D geometry of the helix. They have been explicitly, and to a good approximation, derived analytically under certain assumptions in [2].

As is well known, the differential equation governing a conventional viscoelastic material (i being an integer, e.g. [3–5]) is

$$\sigma + \sum_{i=1}^{I} P_i \frac{d^i\sigma}{dt^i} = E_0\left(\varepsilon + \sum_{j=1}^{J} Q_j \frac{d^j\varepsilon}{dt^j}\right) \tag{2}$$

A more effective fit to experimental data with n and m being significantly smaller than any given I and J is offered by replacing the integer-order derivatives with fractional-order derivatives [6]

$$\sigma + \sum_{i=1}^{n} P_{\alpha_i} D^{\alpha_i}\sigma = E_0\left(\varepsilon + \sum_{j=1}^{m} Q_{\beta_j} D^{\beta_j}\varepsilon\right) \tag{3}$$

Let us note here that the model of type (3)—henceforth, called a *fractional viscoelastic* material—is more consistent with the molecular theories and experiments [4]. In (2) and (3), σ and ε stand for stress and strain, while P and Q are the relaxation and retardation times, respectively, and E_0 is the relaxed magnitude of elastic modulus (prolonged modulus of elasticity). Also, $\alpha_i(i=1,2,\ldots,n)$ and $\beta_j(j=1,2,\ldots,m)$ are the fractional parameters $(0<\alpha_i,\beta_j<1)$ and $D^{\alpha_i}\sigma$ and $D^{\beta_j}\varepsilon$ are the fractional-order derivatives defined as (see e.g. [7])

$$D^{\alpha}f(t) = \frac{\partial^{\alpha}}{\partial t^{\alpha}}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}d\tau \quad 0<\alpha<1 \tag{4}$$

where Γ is the gamma function.

The integrodifferential operator in (4) is of a Caputo-type [6]. It has a fading memory because of the convolution with $t^{-\alpha}$. Here, if $\alpha=1$ we say that the system has a perfect memory and if $\alpha=0$ there is no memory [8]. For any value $0<\alpha<1$ the system has partial memory. The physical interpretation of this operator is interesting: Let us consider δ as a displacement of a simple rod under uniaxial loading. When $\alpha=0$, $D^{\alpha}\delta=\delta$ and when $\alpha=1$, $D^{\alpha}\delta=\dot{\delta}$. Thus, by multiplying a constant parameter by $D^{\alpha}\delta$, depending on the value of α, one may obtain either the elastic force (spring model) or the damping force (dashpot model). For any $0<\alpha<1$, the derived force is a combination of dashpot and spring models. In other words, the range $0<\alpha<1$ is a spectrum of a continuous change from spring to dashpot model.

Due to this property, the fractional model (3) is far more accurate than the model (2). For a wide range of macroscopically homogenous viscoelastic materials including say, elastomers, thermoplastics, and thermostiffening materials, the constitutive equation between stress and strain can be modeled only by using terms up to the first derivatives on the LHS and RHS of Eq. (3); this is called a *5-parameter model*. However, if one wants to model the same material with Eq. (2), many higher integer-order derivative terms on the LHS and RHS must be taken into account in order to achieve a comparable accuracy [9,10].

An interesting mathematical property of the fractional derivative is its Laplace transform [7]

$$L\langle D^{\alpha}f(t)\rangle = s^{\alpha}L\langle f(t)\rangle \quad l-1<\alpha<l \quad l\in \mathrm{N} \tag{5}$$

where

$$L\langle f(t)\rangle = \int_{-\infty}^{\infty} e^{-st}f(t)dt \tag{6}$$

The Caputo definition of a fractional-order derivative has been used in Eq. (4) because, in contradistinction to the Riemann-Liouville definition, it yields zero for a constant. This is why the initial conditions do not appear in Eq. (5).

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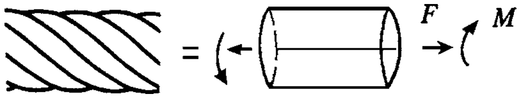

Fig. 1 A system of helical strands, showing a coupling of axial with torsional responses

2 Fractional-Order Equations Governing the Effective Helix' Response

Viscoelastic responses of a helix based on integer-order derivatives were studied in [2,3]. In this note we turn our attention to time dependent response of a helix by using the fractional approach. Considering the viscoelastic fractional-order differential equation (3) as a dilatational response of a material, its Laplace transform becomes

$$\underbrace{\left(1+\sum_{i=1}^{n} P_{\alpha_i} s^{\alpha_i}\right)}_{P(s)}\bar{\sigma}(s) = \underbrace{E_0\left(1+\sum_{j=1}^{m} Q_{\beta_j} s^{\beta_j}\right)}_{Q(s)}\bar{\varepsilon}(s) \tag{7}$$

where the bar sign over σ and ε indicates the parameter in the Laplace transformed domain. $P(s)$ and $Q(s)$ are the coefficients of $\bar{\sigma}(s)$ and $\bar{\varepsilon}(s)$ for the assumed viscoelastic model, and unlike their counterparts in integer-order derivatives, they are no longer polynomials. Instead, they are expressions dependent on the fractional-order power of Laplace parameter. By recourse to the correspondence principle of viscoelasticity, and following a similar approach as that in the case of Eq. (2) in [2], it can be shown that one can recover Hooke's law in the transformed domain from Eq. (7) with the subsequent substitution for the transformed bulk modulus

$$3s\bar{K} = \frac{Q(s)}{P(s)} = \frac{\left(1+\sum_{i=1}^{n} P_{\alpha_i} s^{\alpha_i}\right)}{E_0\left(1+\sum_{j=1}^{m} Q_{\beta_j} s^{\beta_j}\right)} \tag{8}$$

An analogous formula can be derived for the shear response of isotropic materials. Unlike the case of integer-order derivatives, there is no constraint upon initial conditions of stress and strain in the bulk and shear behavior.

Previous experimental tests [10] indicate that most viscoelastic materials can be modeled sufficiently accurately by using only the first fractional derivative terms in each series of Eq. (3): $n=m=1$. We consider such a model for the bulk as well as the shear response of the fractional viscoelastic material making up the helical strand

$$\sigma(t) + PD^{\alpha}\sigma(t) = Q_0\varepsilon(t) + QD^{\alpha}\varepsilon(t) \tag{9}$$

$$S(t) + pD^{\beta}S(t) = q_0\gamma(t) + QD^{\beta}\gamma(t) \tag{10}$$

In the above, as discussed in detail in [8], the fractional-order derivatives on bulk stress (or shear stress) and bulk strain (or shear strain) are assumed to be the same. This reduces the number of parameters in the model from five to four. Considering the dilatational behavior, the Laplace transform of (9) is simply

$$\bar{\sigma}(s)\underbrace{(1+Ps^{\alpha})}_{P(s)} = \underbrace{(E_0+Qs^{\beta})}_{Q(s)}\bar{\varepsilon}(s) \tag{11}$$

so that the transformed modulus will be

$$3s\bar{K} = \frac{Q(s)}{P(s)} = \frac{E_0+Qs^{\beta}}{1+Ps^{\alpha}} \tag{12}$$

Analogous equations apply to the shear response of the helix. By using all these equations in the formulas (16)–(19) of [2], extensively rearranging the terms and applying the inverse Laplace transform to the fractional-order terms, the differential equations of the viscoelastic helix become

$$\Psi_{\sigma}\sigma = \Psi_{\varepsilon}\varepsilon + \Psi_{\tau}\tau \tag{13}$$

$$\Psi'_{\mu}\mu = \Psi'_{\varepsilon}\varepsilon + \Psi'_{\tau}\tau \tag{14}$$

The operators in (13) are

$$\Psi_{\sigma} = [c_0 + c_1D^{\alpha} + c_2D^{\beta} + c_3D^{\alpha+\beta} + c_4D^{2\alpha} + c_5D^{2\beta} + c_6D^{2\alpha+\beta} + c_7D^{\alpha+2\beta} + c_8D^{2\alpha+2\beta} + c_9D^{3\beta} + c_{10}D^{3\beta+\alpha} + c_{11}D^{3\beta+2\alpha}] \tag{15}$$

$$\Psi_{\varepsilon} = [h_0 + h_1D^{\alpha} + h_2D^{\beta} + h_3D^{\alpha+\beta} + h_4D^{2\alpha} + h_5D^{2\beta} + h_6D^{2\alpha+\beta} + h_7D^{\alpha+2\beta} + h_8D^{2\alpha+2\beta} + h_9D^{3\beta} + h_{10}D^{3\beta+\alpha} + h_{11}D^{3\beta+2\alpha}] \tag{16}$$

$$\Psi_{\tau} = [l_0 + l_1D^{\alpha} + l_2D^{\beta} + l_3D^{\alpha+\beta} + l_4D^{2\alpha} + l_5D^{2\beta} + l_6D^{2\alpha+\beta} + l_7D^{\alpha+2\beta} + l_8D^{2\alpha+2\beta} + l_9D^{3\beta} + l_{10}D^{3\beta+\alpha} + l_{11}D^{3\beta+2\alpha}] \tag{17}$$

The operators in (14) are very similar in form (and, therefore, not reproduced for the sake of brevity) but, certainly, have a different set of coefficients.

Several observations are in order here:

(i) Setting $\alpha=\beta=1$ in (9) and (10) converts the fractional model to an integer-order model of Zener type. The operators (15)–(17) reduce to

$$\Psi_{\sigma} = [g_0 + g_1D^1 + g_2D^2 + g_3D^3 + g_4D^4 + g_5D^5] \tag{18}$$

$$\Psi_{\varepsilon} = [e_0 + e_1D^1 + e_2D^2 + e_3D^3 + e_4D^4 + e_5D^5] \tag{19}$$

$$\Psi_{\tau} = [b_0 + b_1D^1 + b_2D^2 + b_3D^3 + b_4D^4 + b_5D^5] \tag{20}$$

which coincides with the results obtained in [2].

(ii) The operators (15)–(17) are asymmetric with respect to α and β; there are some extra terms that involve higher order derivatives of β (which was the fractional exponent for the shear response at the material level). This indicates that in the constitutive equations of viscoelastic helix, the effect of shear modulus is more pronounced than that of the bulk modulus. Such an effect does not arise in the integer-order models for the helix; operators in (18)–(20) involve six terms each.

(iii) Another interesting feature is that, even though the chosen fractional models at the material level are such that $0<\alpha$, $\beta<1$, the order of the fractional derivatives in the helix is definitely higher and can be, in general, greater than 1. Note that in real materials $0<\alpha$, $\beta<1$, as exemplified by α, $\beta \cong 0.5$ for elastomers [7,8].

We postulate that the particular arrangement of the helix geometry through the bending constraint of 3D helical segments or other types of constraints such as the compatibility of axial-torsional deformations, explain the observations (ii) and (iii).

3 Relaxation Response of the Helix

In general, the advantage of having the governing differential equations of a viscoelastic helix (either integer or fractional type) is that one can simply study the macroscopic behavior of helices, for instance damped vibrations of such helical elements [11–14], or macroscopic creep/relaxation phenomena which we examine here. The applications can range from cables used in suspension bridges or in prestressed concrete girders, to biological tissues which involve helical geometries.

Let us now focus on the governing equations (13) and (14) and the operators (15)–(18) to find the relaxation response of the helix. Since in the relaxation test the strains are constant with time, their fractional-order, Caputo-type derivatives become zero. Henceforth the operators (16) and (17) reduce to constants as

$$\Psi_{\varepsilon} = h_0 \tag{21a}$$

$$\Psi_{\tau} = l_0 \tag{21b}$$

while (15) remains unchanged. With this simplification, one can take the Laplace transform of Eq. (15) to find

$$\bar{\sigma}(s) = \frac{h_0\varepsilon + l_0\tau}{s\Psi(s)} \tag{22}$$

where $\Psi(s)$ is an algebraic operator

$$\Psi(s) = c_0 + c_1 s^{\alpha} + c_2 s^{\beta} + c_3 s^{\alpha+\beta} + c_4 s^{2\alpha} + c_5 s^{2\beta} + c_6 s^{2\alpha+\beta} + c_7 s^{\alpha+2\beta} + c_8 s^{2\alpha+2\beta} + c_9 s^{3\beta} + c_{10} s^{3\beta+\alpha} + c_{11} s^{3\beta+2\alpha} \tag{23}$$

The inverse transform of Eq. (22) exists, is real, continuous and causal, see the Appendix. Thus, the time dependent stress is given by (A8). In that equation, m is the smallest common denominator of the exponents of s in $\Psi(s)$. Note that the macroscopic relaxation response contains three terms. The first term is a constant independent of time; it is indeed the smallest possible value of $\sigma(t)$ (because the other two terms vanish as t approaches infinity). Note here that the structure of a 5-parameter fractional model (regardless of its fading memory) is similar to the integer-order Zener model. Hence, there is a spring (even though it might be weaker in the fractional case as opposed to integer case) parallel to the dashpot that prevents the total stress from approaching zero as time goes to infinity. Furthermore, the dashpot itself has a partial memory and can also behave like a spring. The second term is an integral that decreases with increasing time, while the last term is a sum of exponentially decaying sinusoidal functions. A similar expression can be found for the time dependent couple-stress $\mu(t)$ in relaxation.

In creep phenomena, the stress remains constant and strains increase in time. Therefore, Ψ_{σ} reduces to c_0 while Ψ_{ε} and Ψ_{τ} remain unchanged as a sum of fractional derivative operators. In this case, one cannot use either of Eqs. (13) or (14) to find the time dependent strains, but the coupled systems of Eqs. (13) and (14) have to be tackled. This again is possible analytically by making use of the Laplace transform and the reside theorem given in the Appendix. It turns out that, also in the creep test, there is a constant term in the solution, indicating that strains have an upper bound as time goes to infinity. As the resulting expressions are very lengthy, we do not show them.

4 Conclusions

The effective (along-the-axis) response of helices made of viscoelastic materials is far from trivial. In essence, the constitutive equations of helices are more complex than those of their constituents. The effect was brought out earlier for materials with integer-order derivatives [2] and, as shown here, is even stronger in the case of fractional-order derivatives. Overall, the effect is due to the 3D geometry of the helix.

More specifically, the influence of shear modulus of the fractional viscoelastic material is more complicated and dominant than that of its bulk modulus on the effective response of the helix. This observation is valid only when considering fractional-order derivatives for the helix material. In the special case of integer-order derivatives, one recovers the conventional differential equations. Note here that one cannot obtain the equations of the helix made of a fractional viscoelastic material by a direct generalization of those of the helix made of a conventional viscoelastic material.

The fractional-order equations governing the effective helix' response are employed to derive the explicit analytic solution in the standard relaxation test. In summary, the results of this paper provide guidance on equations and responses of 3D chiral fractional viscoelastic materials, similar to those on thermoelastic helices [15] which offered guidance on 3D chiral thermoelastic materials.

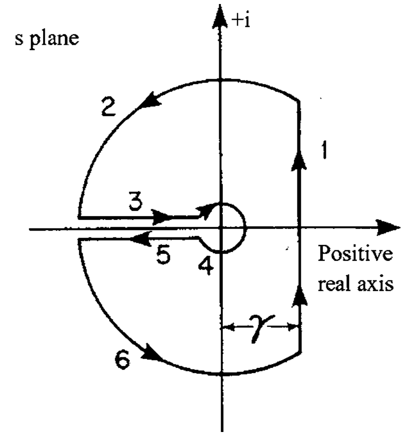

Fig. 2 Integration contour in the *s* plane, after [7]

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Appendix

One can convert the fractional expression $\Psi(s)$ to a polynomial of integer order by

$$\Psi(s) = \sum_{J=1} b_j s^{j/m} = \sum_{J=1} b_j u^j = X(u) \tag{A1}$$

where $u = s^{1/m}$ and m is the smallest common denominator of the fractional exponents of s in $\Psi(s)$. Clearly, some of the coefficients b_j are zero, while any nonzero b_j corresponds to the coefficient of s in $\Psi(s)$ whose exponent becomes equal to j/m. The inverse transform of $\bar{\sigma}(s)$ exists and is real, continuous and causal when (i) $\bar{\sigma}(s)$ is analytic for $\mathrm{Re}(s) > 0$, (ii) $\bar{\sigma}(s)$ is real for s real and positive, (iii) $\bar{\sigma}(s)$ is of order $s^{-\gamma}$, where $\gamma > 1$, for $|s|$ large in the right half of the s-plane [13]. One can simply show that $\bar{\sigma}(s)$ satisfies all these three conditions. The inverse Laplace transform of (22) is then

$$L^{-1}[\bar{\sigma}(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}(\bar{\sigma}(s))ds \tag{A2}$$

which can be evaluated by extending the line integral into a closed contour integration as in Fig. 2.

Next, we recall the residue theorem, which states that the integral along any closed contour, divided by $2\pi i$, is equal to the sum of the residues of poles of the integrand within that contour. In Fig. 2, the contour is divided into six segments with arrows which indicate the direction of integration. Note that, since the branch cut of $s^{1/m}$ is along the negative real axis of the s plane, segments 3–5 are required here. Using the residue theorem, we can write

$$\frac{1}{2\pi i}\int_{1} e^{st}(\bar{\sigma}(s))ds = -\frac{1}{2\pi i}\sum_{k=2}^{6}\int_{k} e^{st}(\bar{\sigma}(s))ds + \sum_{j} b_j \quad \text{(A3)}$$

Equation (A2) is the left-hand side of Eq. (A3) when its limits are extended infinitely in the negative and positive imaginary directions of the s plane. To ensure the continuity of the closed contour, the radii of segments 2 and 6 are increased infinitely and consequently segments 3 and 5 are stretched to infinity on the negative real axis. It can be shown that the integrals along contours 2 and 6 are zero when the radius approaches infinity. The contour integral along the segment 4 can be obtained by using the following lemma (MacRobert [14]):

"If $\lim_{s\to a}\{(s-a)f(s)\}=k$, where k is a constant, then $\lim_{s\to a}\{\int f(s)ds\}=i(\theta_2-\theta_1)k$, the integral being taken for $s\to a$ and $r\to 0$ around an arc from θ_1 to θ_2 of the circle $|s-a|=r$."

It follows that

$$\lim_{s\to 0}\{s\bar{\sigma}(s)\} = \frac{h_0\varepsilon + l_0\tau}{c_0} \quad \text{(A4)}$$

so that, the contour integration along segment 4 becomes

$$\int_{4}\bar{\sigma}(s)ds = -2\pi i\left(\frac{h_0\varepsilon + l_0\tau}{c_0}\right) \quad \text{(A5)}$$

where $(\theta_2-\theta_1)=-2\pi$ by convention. One can also show

$$\int_{3} e^{st}(\bar{\sigma}(s))ds + \int_{5} e^{st}(\bar{\sigma}(s))ds = -2i\,\mathrm{Im}\int_{0}^{\infty} e^{-rt}(\bar{\sigma}(re^{-i\pi}s))dr \quad \text{(A6)}$$

Using a conventional technique, the residues are then calculated as

$$b_j = \lim_{s\to\lambda_j^m}\{(s-\lambda_j^m)(\bar{\sigma}(s)e^{st})\} \quad \text{(A7)}$$

Here λ_j refers to the jth root of the integer polynomial $X(u)$. In view of Eqs. (A1) and (22), the roots of $X(u)$ correspond to the poles of $\bar{\sigma}(s)$ involving fractional exponents. Note that $s=0$ is not included within the closed contour of Fig. 2 and, therefore, its residue is not required.

Finally, by adding (A5)–(A7), Eq. (A3) becomes

$$\sigma(t) = \frac{h_0\varepsilon + l_0\tau}{c_0} + \frac{1}{\pi}\,\mathrm{Im}\left[\int_0^{\infty}\bar{\sigma}(re^{-i\pi})e^{-rt}dr\right] + (h_0\varepsilon + l_0\tau)\sum_j (s - \lambda_j^m)\frac{e^{\lambda_j^m t}}{\lambda_j^m\Psi(\lambda_j^m)} \quad \text{(A8)}$$

The summation in the last term is over the residues that are included within the closed contour in Fig. 2. The poles of Eq. (22) λ_j [those that make $\Psi(s)=0$] were found in the $s^{1/m}$ plane. As the Laplace transform is performed in the s plane, the poles λ_j should be transformed into λ_j^m so as to be on the s plane. That transformation, however, causes some of the original poles to be mapped onto Riemann surfaces not included within the closed contour of integration in the s plane. According to the residue theorem, the residues of such poles do not contribute to the solution. Thus, the summation over the index j in Eq. (A8) applies to those poles only that remain in the closed contour's plane after the transformation λ_j^m. The reside theorem in conjunction with fractional-order derivatives has been used in [7] in a slightly different way.

References

[1] Samras, R. K., Skop, R. A., and Milburn, D. A., 1974, "An Analysis of Coupled Extensional-Torsional Oscillations in Wire Rope," ASME J. Eng. Ind., **96**, pp. 1130–1135.
[2] Shahsavari, H., and Ostoja-Starzewski, M., 2005, "On Elastic and Viscoelastic Helices," Philos. Mag., **85**(33–35), pp. 4213–4230.
[3] Conway, T. A., and Costello, G. A., 1993, "Viscoelastic Response of a Strand," ASME J. Appl. Mech., **60**(2), pp. 534–540.
[4] Bagley, R. L., and Torvik, P. J., 1983, "A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity," J. Rheol., **27**(3), pp. 201–210.
[5] Bland, D. R., 1960, *Theory of Linear Viscoelasticity*, Pergamon, New York.
[6] Caputo, M., 1967, "Linear Models of Dissipation Whose *Q* is Almost Frequency Independent—Part II," Q. J. R. Astron. Soc., **13**, pp. 529–539.
[7] Bagley, R. L., and Torvik, P. T., 1983, "Fractional Calculus—A Different Approach to the Analysis of Viscoelastically Damped Structures," AIAA J., **21**(5), pp. 741–748.
[8] Baker, W. P., Eldred, L. B., and Palazotto, A., 1996, "Viscoelastic Material Response With a Fractional-Derivative Constitutive Model," AIAA J., **34**(3), pp. 596–600.
[9] Beda, T., and Chevalier, Y., 2004, "Identification of Viscoelastic Fractional Complex Modulus," AIAA J., **42**(7), pp. 1450–1456.
[10] Rossikhin, Y. A., and Shitikova, M. V., 2004, "Analysis of the Viscoelastic Rod Dynamics via Models Involving Fractional Derivatives or Operators of Two Different Orders," Shock Vib. Dig., **36**(1), pp. 3–26.
[11] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J., 2006, *Theory and Application of Fractional Differential Equations*, Elsevier, New York.
[12] Shahsavari, H., and Wihler, T. P., 2006, "Viscoelastodynamics of Helices," in preparation.
[13] Churchill, R. V., 1958, *Book of Operational Mathematics*, 2nd ed., McGraw-Hill, New York.
[14] MacRobert, T. M., 1962, *Functions of a Complex Variable*, 5th ed., Macmillan, New York.
[15] Ostoja-Starzewski, M., 2003, "Thermoelastic Waves in a Helix With Parabolic or Hyperbolic Heat Conduction," J. Therm. Stresses, **26**, pp. 1205–1219.