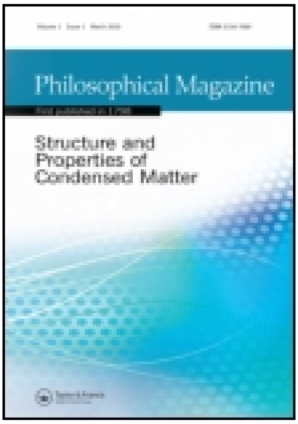

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On elastic and viscoelastic helices

H. Shahsavari [a] & M. Ostoja-Starzewski [a b]
[a] Department of Mechanical Engineering, McGill University, 817 Sherbrooke Street West, Montréal, Québec, H3A 2K6, Canada
[b] McGill Institute for Advanced Materials, McGill University, 817 Sherbrooke Street West, Montréal, Québec, H3A 2K6, Canada
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On elastic and viscoelastic helices

H. SHAHSAVARI† and M. OSTOJA-STARZEWSKI*†‡

†Department of Mechanical Engineering, McGill University,
817 Sherbrooke Street West, Montréal,
Québec, H3A 2K6 Canada
‡McGill Institute for Advanced Materials, McGill University,
817 Sherbrooke Street West, Montréal,
Québec, H3A 2K6 Canada

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Under consideration are helices made of linear elastic and viscoelastic materials, henceforth called elastic and viscoelastic helices. While the effective (macro level) mechanical response of any helix exhibits coupling of axial with torsional responses, of interest in this study is the derivation of that response from mechanics of a single helical strand (micro level). First, using the earlier results of Costello [*Theory of Wire Rope* (Springer, New York, 1997)], we develop explicit forms of all the effective constitutive coefficients of a linear elastic helix and plot their dependencies on the geometry and material constants of the strand. Next, using the correspondence principle of viscoelasticity, we derive differential equations of a helix at the macro level by considering three types of viscoelastic models of the strand: Kelvin, Maxwell, and Zener. In general, the helix macro level linear viscoelastic response is different in type (and more complex) from that of the viscoelastic material at the micro level. It is only in the singular case of the strand material's Poisson ratio equal to zero that the type of viscoelastic response is qualitatively the same (i.e. governed by the same order differential equation) as the viscoelastic response of the strand at the micro level. Consequently, direct viscoelastic generalizations of effective constitutive equations of helices, not based on analyses such as those presented here, are likely to be invalid.

1. Background

Helically wound fibres or wires constitute a wide class of important engineering components. It is well known that a major advantage of such elements is their capacity to support large axial loads with comparatively small bending or torsional stiffness. Some of the applications often require a quantitative evaluation of the relevant mechanical parameters. Two important fields of application are cables and overhead electrical conductors.

In recent years, considerable progress has been made in the development of models to predict the response of a helix or any helically wound bundle like a wire rope. Since there are several parameters that may vary in the construction of a helix, such models can be used to determine the effects of possible variations

*Corresponding author. Email: martin.ostoja@mcgill.ca

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of the parameters on the performance of a rope. In these models, the underlying geometry is generally that of a core strand surrounded by one or several helical strands. Each field has developed a specific body of knowledge based on the previous work and extensive testing experience, leading to useful rules for particular practical applications. A review, and then a book on the subject of helical strand models were published by Costello [1, 2]. As indicated by the book's title, the setting was the wire rope applications. Another, more recent review has been published by Cardou and Jolicoeur [3].

In the overhead electrical conductor technology, the investigations are mostly oriented towards bending vibration phenomena. In these studies, the conductor is simply treated as a taut string having some bending stiffness and damping coefficients, which have to be found experimentally.

Indeed, most of the research done so far on helices and helical wires has been restricted to linear elastic material behaviour, and only a limited amount on inelastic behaviour — primarily experimental in character — have been reported. The main objective of the present paper is to derive an effective response of a viscoelastic helix, or a bundle of helices, from micromechanics. To set the stage, in section 2, we first derive such explicit formulas in the simpler case when the helix (or a helical bundle) is linear elastic. To determine the effective viscoelastic response of helices, as shown in section 3, we assume certain viscoelastic models at the micro level for the helix material. Then, by using the elastic-viscoelastic correspondence principle, we derive the effective constitutive differential equations of the helix at the macro level.

Our focus on a micro-macro bridge for helices should be a fitting tribute to Professor G.A. Maugin, whose researches have revolved around and motivated others to work on such topics for many decades. In what follows, we concentrate on the available strand models, irrespective of their field of application. Thus, a helical system is called a bundle — generally composed of a core and several helical strands — independent of any applied connotation.

An exact analytical determination of the mechanical behaviour of a helical bundle from its microgeometry and Hooke's law of the helix material is very difficult if not impossible [2]. Several approximations and assumptions have to be made to render an analytical solution more tractable. However, as shown in section 2, one can derive an approximate formula for the effective response of the helix, whose principal characteristic is the coupling of axial with torsional responses.

Henceforth, for convenience, by a helix we understand either a single strand or a bundle consisting of a centre strand and m outer strands, such as in a typical wire. With reference to figure 1, we work under the following assumptions:

(i) The bundle is composed of strands, all uniformly spaced along the perimeter of a circle of radius $r_2 = R_1 + R_2$, thus forming a ring in a plane perpendicular to the bundle's axis without touching each other.
(ii) Each strand's equilibrium configuration is a helix of constant radius $r_2 = R_1 + R_2$ and constant helix angle α_2 (the subscript 2 refers to the deformed configuration).
(iii) Strands are linear elastic (with an axial modulus and Poisson's ratio) and undergo very small strains.
(iv) Friction and contact deformations are neglected.

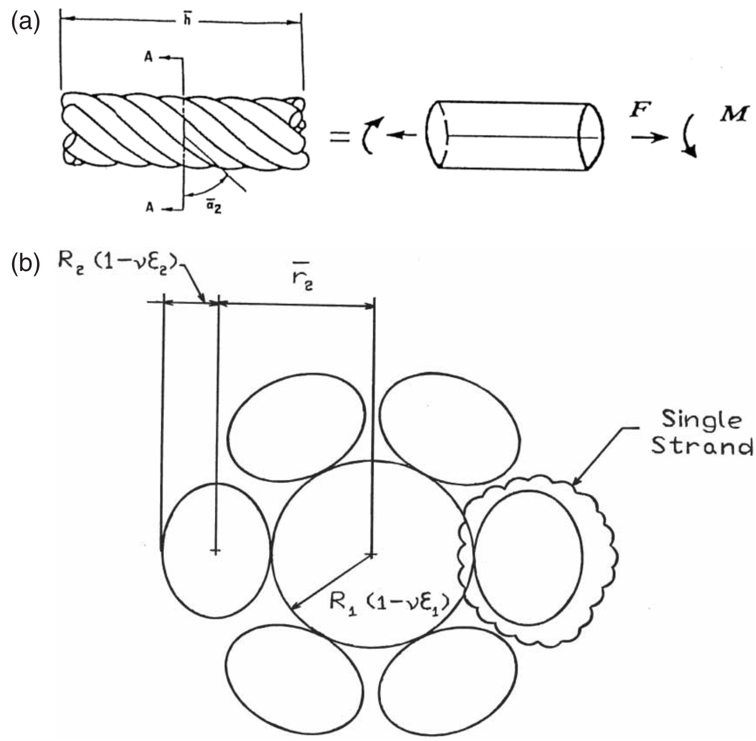

Figure 1. (a) Bundle of helically wound strands, and their mechanical model. (b) Cross-section A–A consisting of six outer helical strands and one inner (straight) strand.

The axial strain of a bundle is defined as

$$\varepsilon = \frac{\bar{h} - h}{h} \tag{1}$$

where h and $\bar{h}$ are the original and final lengths of the bundle, respectively (figure 1). The rotational strain of the bundle is defined as

$$\beta_2 = r_2 \frac{\bar{\theta} - \theta}{h} \tag{2}$$

where θ and $\bar{\theta}_2$ are the initial and final angles that a bundle sweeps out in a plane perpendicular to the bundle's axis, respectively. Finally, the angle of twist per unit length of the bundle is defined as (note $\beta_2 = r_2\tau_2$)

$$\tau_2 = \frac{\bar{\theta} - \theta}{h}. \tag{3}$$

The basic equations of a linear elastic helix can be written as [4]

$$\frac{F}{AE} = C_1\varepsilon + C_2\tau \qquad \frac{M}{ER^3} = C_3\varepsilon + C_4\tau \tag{4a,b}$$

where

$$F = F_1 + F_2 \qquad M = M_1 + M_2 \qquad R = R_1 + 2R_2 \tag{5a,b,c}$$

and F_1 and M_1 are the contributions of the centre strand and F_2 and M_2 are the contributions of the outer strands. In this paper we first focus on the explicit formulas of four constitutive coefficients appearing in (4a,b) in terms of actual helical geometry and elastic properties of strands, and then on their parameter dependencies. We then use these formulas to derive effective differential-type equations of helices, when strands are made of either Kelvin, or Maxwell or Zener viscoelastic materials. Our principal interest is in determining whether a strand of, say, a Kelvin material will result in an effective Kelvin-type, or a more complex helix.

2. Elastic helix

2.1. *Explicit forms of the constitutive coefficients*

In order to find all the C_i coefficients in (4a,b), we focus on a bundle, and assume $\beta = R\tau_s = 0$ and $\varepsilon = \varepsilon_1$. Then, by following the approach of Costello [2], we obtain

$$\frac{F}{AE} = C_1\varepsilon_1 \qquad \frac{M}{ER^3} = C_3\varepsilon_1 \tag{6a,b}$$

where

$$C_1 = \frac{1}{AE}\left(\pi E R_1^2 + \frac{m}{\varepsilon_1}\left[T_2\sin\alpha_2 + N_2'\cos\alpha_2\right]\right) \tag{7}$$

$$C_3 = \frac{m}{\varepsilon_1 E R^3}\left[H_2\sin^3\alpha_2 + G_2'\cos\alpha_2\left(1+\sin^2\alpha_2\right) + T_2 r_2\cos\alpha_2\right]. \tag{8}$$

In the above m is the number of outer strands, N_2' is the shear force on a strand's cross-section, T_2 is the axial tension in the strand; G_2' is the bending moment on a strand cross-section; H_2 is the twisting moment in the strand. The subscript 2 refers to the outer strands, while the prime indicates a component perpendicular to the strands' axis.

Focusing the analysis on the outside strand, working from the equilibrium equations for a single helical strand, we find the following formulas:

$$\varepsilon_1 = \varepsilon_2 + \frac{\Delta\alpha_2}{\tan\alpha_2} \qquad \beta_2 = r_2\tau_s = \frac{\varepsilon_2}{\tan\alpha_2} - \Delta\alpha_2 + \nu\frac{(R_1\varepsilon_1 + R_2\varepsilon_2)}{r_2\tan\alpha_2} \tag{9a,b}$$

$$G_2' = \frac{\pi}{4}R_2\Delta\kappa_2' E R_2^3 \qquad H_2 = \frac{\pi}{4(1+\nu)}R_2\Delta\tau_2 E R_2^3 \qquad T_2 = \pi\varepsilon_2 E R_2^2 \tag{10a,b,c}$$

$$N'_2 = \frac{H_2}{R_2}\frac{\cos^2\alpha_2}{r_2/R_2} - \frac{G'_2}{R_2}\frac{\sin\alpha_2\cos\alpha_2}{r_2/R_2} \tag{11}$$

$$R_2\Delta k'_2 = -\frac{2\sin\alpha_2\cos\alpha_2}{r_2/R_2}\Delta\alpha_2 + \nu\frac{(R_1\varepsilon_1 + R_2\varepsilon_2)}{r_2}\frac{\cos^2\alpha_2}{r_2/R_2} \tag{12}$$

$$R_2\Delta\tau_2 = \frac{\left(1 - 2\sin^2\alpha_2\right)}{r_2/R_2}\Delta\alpha_2 + \nu\left(\frac{R_1\varepsilon_1 + R_2\varepsilon_2}{r_2}\right)\frac{\sin\alpha_2\cos\alpha_2}{r_2/R_2}. \tag{13}$$

Here ε_1 is the axial strain in the centre strand ($\varepsilon = \varepsilon_1$) while ε_2 is the axial strain in the outer strand. Also, k'_2 is the curvature of the outer strand and τ_2 is the twist per unit length of the outer strand. Considering the assumption $\beta = R\tau_s = 0$ ($\Rightarrow \tau_s = 0$) and $\varepsilon = \varepsilon_1$, equations (9a,b) become

$$\varepsilon_1 = \varepsilon_2 + \frac{\Delta\alpha_2}{\tan\alpha_2} \qquad 0 = \frac{\varepsilon_2}{\tan\alpha_2} - \Delta\alpha_2 + \nu\frac{(R_1\varepsilon_1 + R_2\varepsilon_2)}{r_2\tan\alpha_2}. \tag{14a,b}$$

In view of (14a,b), we obtain

$$\varepsilon_2 = \frac{\left(r_2\tan^2\alpha_2 - \nu R_1\right)}{r_2 + r_2\tan^2\alpha_2 + \nu R_2} \qquad \Delta\alpha_2 = \left(\frac{r_2(1+\nu)}{r_2 + r_2\tan^2\alpha_2 + \nu R_2}\right)\varepsilon_1\tan\alpha_2. \tag{15a,b}$$

Finally, substituting (15a,b) into equations (10) through (13), and taking $\beta_2 = r_2\tau_s$ equations (7) and (8) become

$$\begin{aligned} C_1 = {} & \frac{R_1^2}{\left(R_1^2 + mR_2^2\right)} + \frac{mR_2^2}{\left(R_1^2 + mR_2^2\right)\left(r_2 + r_2\tan^2\alpha_2 + \nu R_2\right)} \\ & \times \left[\left(r_2\tan^2\alpha_2 - \nu R_1\right)\sin\alpha_2 + \frac{R_2^2\left(1 - 2\sin^2\alpha_2\right)\sin\alpha_2\cos^2\alpha_2}{4r_2}\right. \\ & \left. + \frac{R_2^2\sin^3\alpha_2\cos^2\alpha_2(1+\nu)}{2r_2} - \frac{\nu^2R_2^2\sin\alpha_2\cos^4\alpha_2\left(R_1 + r_2\tan^2\alpha_2\right)}{4(1+\nu)r_2^2}\right]. \end{aligned} \tag{16}$$

$$\begin{aligned} C_3 = {} & \frac{m\pi R_2^2}{(R_1 + 2R_2)^3\left(r_2 + r_2\tan^2\alpha_2 + \nu R_2\right)} \\ & \times \left[\frac{\left(1 - 2\sin^2\alpha_2\right)R_2^2\tan\alpha_2\sin^3\alpha_2}{4} + \frac{\nu R_2^2}{4(1+\nu)r_2}\left(R_1 + r_2\tan^2\alpha_2\right)\sin^4\alpha_2\cos\alpha_2\right. \\ & - \frac{(1+\nu)}{2}R_2^2\sin^2\alpha_2\cos\alpha_2\left(1 + \sin^2\alpha_2\right) + \frac{\nu R_2^2}{4r_2}\left(R_1 + r_2\tan^2\alpha_2\right) \\ & \left. \times \left(1 + \sin^2\alpha_2\right)\cos^3\alpha_2 + \left(r_2\tan^2\alpha_2 - \nu R_1\right)r_2\cos\alpha_2\right]. \end{aligned} \tag{17}$$

To determine C_2 and C_4, assume $\varepsilon = 0$ and $\beta = \beta_2 = r_2\tau_s$. Following a similar procedure as for C_1 and C_3, we find

$$C_2 = \frac{m r_2 R_2^2}{(R_1 + 2R_2)\left(R_1^2 + mR_2^2\right)\left(r_2 + r_2\tan^2\alpha_2 + \nu R_2\right)} \times \Bigg[r_2 \tan\alpha_2 \sin\alpha_2 + \frac{R_2^2}{4(1+\nu)r_2}\left(\left(2\sin^2\alpha_2 - 1\right)\sin^2\alpha_2\cos\alpha_2\right) - \frac{R_2^2}{2r_2}\sin^4\alpha_2\cos\alpha_2 - \frac{\nu^2 R_2^3 \sin^2\alpha_2 \cos^3\alpha_2}{4r_2^2(1+\nu)} \Bigg]. \tag{18}$$

$$C_4 = \frac{\pi R_1^4}{4(R_1 + 2R_2)^4(1+\nu)} + \frac{r_2 \pi m R_2^2}{(R_1+2R_2)^4\left(r_2 + r_2\tan^2\alpha_2 + \nu R_2\right)} \times \Bigg[\frac{R_2^2 \sin^3\alpha_2 \tan^2\alpha_2\left(2\sin^2\alpha_2 - 1\right)}{4(1+\nu)} + \frac{R_2^2\left(1+\sin^2\alpha_2\right)\sin^3\alpha_2}{2}\frac{\nu R_2^3 \sin^5\alpha_2}{4(1+\nu)r_2} + \frac{\nu R_2^3 \sin\alpha_2\left(1+\sin^2\alpha_2\right)\cos^2\alpha_2}{4r_2} + r_2^2\sin\alpha_2 \Bigg]. \tag{19}$$

Note that equations (16) through (19) are quite general. For example, by setting $m = 2$ and neglecting the first terms in C_1 and C_4 (which are the contributions of the core), one may obtain the constitutive coefficients of a system consisting, say, of two helically wound strands.

2.2. *Parametric studies of constitutive coefficients*

In this section we quantitatively assess the dependence of the helix constitutive coefficients (16)–(19) on its geometric and material properties. In what follows, one parameter is chosen to vary at a time with all the other parameters being kept constant.

Note that, in order to compare C_2 and C_3, these two coefficients have now been pre-multiplied by $\pi(R_1^2 + mR_2^2)$ and$(R_1 + 2R_2)^2$, respectively. With this modification, and in light of the Betti–Maxwell reciprocity theorem applied to the helix on the macro level, C_2 and C_3 become equal. However, due to the assumptions involved in their derivation [2], they are not exactly equal, and this is shown in figures 2–4. For the Poisson ratio between -0.75 and 0.5, the fraction C_3/C_2 is 0.93 or higher. Figure 3 shows that, when α is less than about 0.4 rad, C_3 is less than C_2, and higher for other values of α. Also, as α approaches $\pi/2$ (no helical effect), C_3/C_2 gets close to 1, albeit both C_2 and C_3 then approach zero.

Figure 4 shows the dependence of all four helix coefficients on the helix angle α at $\nu = 0.25$. An analogous dependence of strain energies involved in the C_i coefficients is shown in figure 5. The following observations are made here:

(i) When α approaches $\pi/2$, the bundle is a parallel system of straight rods, and the coupling coefficients C_2 and C_3 tend to zero. Also then, C_1 approaches its maximum value, since all the strands are straight (no reduction in the axial capacity).

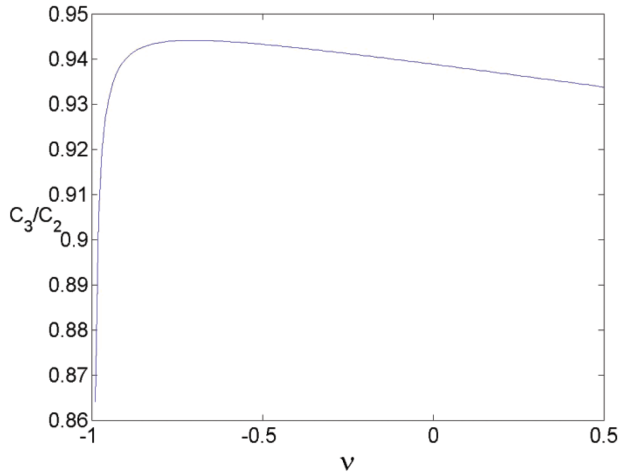

Figure 2. Dependence of C_3/C_2 on the Poisson ratio ν at helix angle $\alpha = 82.5°$.

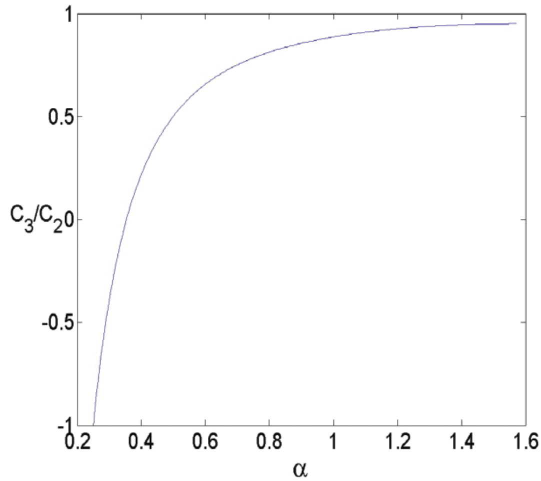

Figure 3. Dependence of C_3/C_2 on the helix angle α at $\nu = 0.25$.

(ii) When α approaches $\pi/2$, C_4 tends to a number proportional to the shear modulus G. That property, however, attains its maximum value at α equal about 0.6 or 0.7 rad. This is because it is much easier to rotate a bundle of parallel strands than helically wound strands, or, in other words, the torsional stiffness of a curved beam is larger than that of a straight beam.

(iii) The maximum effect of the coupling coefficients C_2 and C_3 occurs for α around 1 rad (or 57°). This value corresponds to the inflection point of the C_1 plot. Note that the helix angles, which maximize the helix coefficients, are identical for different constant parameters.

In figure 5 strain energies involved in C_i coefficients (derived from Costello's assumptions) are plotted as functions of α. These energies are based on $\sigma^2/2E$

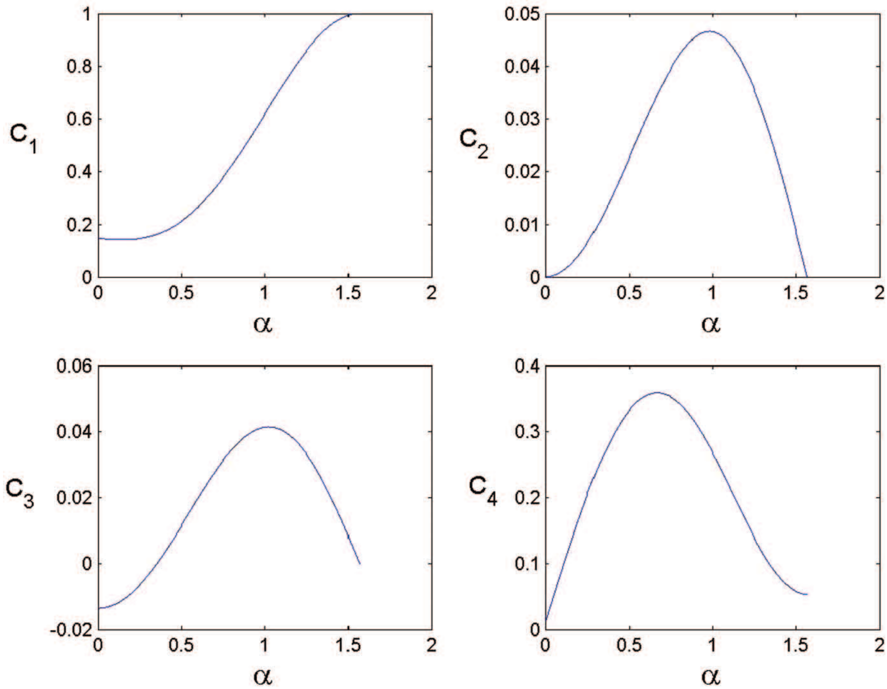

Figure 4. Dependence of four constitutive coefficients C_i of the helix of equation (4a,b) on the helix angle α at $\nu = 0.25$.

and $\tau^2/2G$ as strain energy densities for axial and torsional deformations, and are normalized by dividing by E. Figure 5 shows that

(i) Strain energy plots involved in C_1 and C_4 as functions of α, are qualitatively similar to those of C_1 and C_4 as functions of α (figure 4). As expected, with α approaching $\pi/2$, the energy in the C_1 term tends to a maximum. However, the maximum energy in the C_4 term occurs at a value of α different from $\pi/2$.

(ii) When α is greater than about 0.5 radians, the strain energy plots involved in coupling terms are qualitatively the same. There are some discrepancies between the energies of C_2 and C_3 when α is less than 0.5 rad. This goes back to the fact that C_2 and C_3 used in energy formulas are not exactly equal. But as expected, both energies of coupling terms approach zero when α approaches $\pi/2$.

(iii) The maximum difference between the strain energies involved in C_2 and C_3 terms is about 20%.

3. Viscoelastic helices

3.1. *Application of the correspondence principle*

In most viscoelastic materials, Poisson's ratio (PR) is not a constant, but rather a function of time or frequency [5, 6]. There exist very few special materials with time-independent PRs; for example, when the bulk modulus approaches

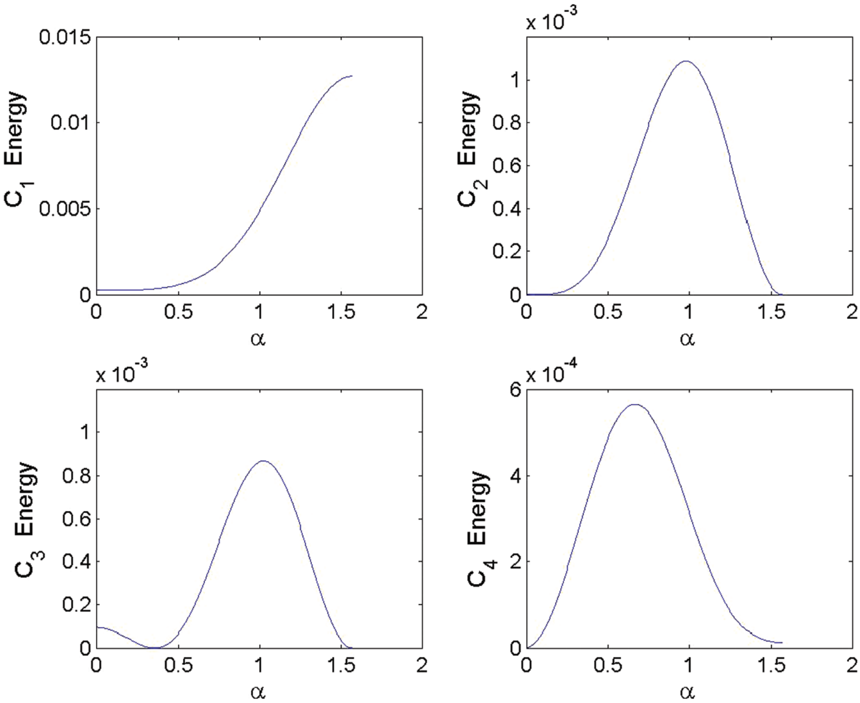

Figure 5. Strain energies involved in the C_i terms as functions of α at $\nu = 0.25$.

infinity. However, most of the practical problems including composites or sandwich structures or problems involving thermal and chemical expansions, such as curing and manufacture of viscoelastic composites, are generally characterized by time-dependent PRs. Thus, for the sake of generality, in this paper we consider viscoelastic helices with time-dependent PRs. First, we briefly recall the concepts and basic equations of the correspondence principle.

As is well known, the elastic stress–strain relations for an isotropic material are

$$\sigma_{kk} = 3K\varepsilon_{kk} \qquad S_{ij} = 2G\varepsilon_{ij} \tag{20a,b}$$

where K and G are the bulk and shear moduli. For simplicity of the notation, in the sequel we do not show kk and ij indices. The integral forms of the viscoelastic stress–strain relations are [7]

$$\sigma(t) = \int_{-\infty}^{t} 3K(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}\,d\tau \qquad S(t) = \int_{-\infty}^{t} 2G(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}\,d\tau. \tag{21a,b}$$

The Laplace transforms of equations (21a,b) are

$$\bar{\sigma}(s) = 3s\bar{K}(s)\bar{\varepsilon}(s) \qquad \bar{S}(s) = 2s\bar{G}(s)\bar{\varepsilon}(s). \tag{22a,b}$$

Here, s is the Laplace parameter and the bar sign on a parameter indicates the parameter in the Laplace space. By comparing equations (20a,b) to equations (22a,b), it turns out that the viscoelastic stress–strain relations for a problem in the Laplace space can be derived from a corresponding elastic problem by replacing

$$K \to s\bar{K}(s) \qquad G \to s\bar{G}(s). \tag{23a,b}$$

The above transformation is sometimes called Carson transform or S-multiplied transform, and is in fact the essence of the corresponding principle.

We now consider the helical strand(s) to be made of a viscoelastic material, described by a differential equation of this general form

$$\sigma + P_1\dot{\sigma} + P_2\ddot{\sigma} + \cdots = Q_0\varepsilon + Q_1\dot{\varepsilon} + Q_2\ddot{\varepsilon} + \cdots. \tag{24}$$

Here the number of dots indicates the order of differentiation with respect to time, while $P_1, P_2, \ldots\ Q_0, Q_1, \ldots$ are the constitutive constants.

Our objective is to determine differential equations governing the helix at the macro level — analogous to (4a,b) pertaining to a linear elastic helix — assuming it is made of strand of a specific viscoelastic model (e.g. Zener type). The elastic helix of section 2 provides a stepping-stone in this respect, and, although we do not have perfect formulas relating macro level response to the micro level strand properties and geometry, the equations (16)–(19) are the best model available to us.

While proceeding by a direct method would be very unwieldy, the correspondence principle of viscoelasticity offers a more convenient path. As all of our elastic helix equations are derived in terms of E and ν, before invoking the correspondence principle, we first recall these classical elasticity relations

$$E = \frac{9KG}{3K+G} \qquad \nu = \frac{3K-2G}{2(3K+G)} \tag{25a,b}$$

and then use Carson transforms of equations (23a,b). This method is very general and straightforward for problems involving time-independent PR.

Another form of the correspondence principle [7] states: "*elastic solutions can be converted to Laplace transformed viscoelastic solutions through the replacement of the elastic moduli and elastic Poisson's ratio by the transform parameter multiplied transforms of the appropriate viscoelastic relaxation functions and viscoelastic Poisson's ratio*." We use this idea and substitute

$$E \to s\bar{E}(s) \qquad \nu \to s\bar{\nu}(s) \tag{26a,b}$$

in equations (16)–(19). Also, Carson transforms of equations (25a,b) give

$$\bar{E}(s) = \frac{9\bar{K}(s)\bar{G}(s)}{3\bar{K}(s)+\bar{G}(s)} \qquad \bar{\nu}(s) = \frac{3\bar{K}(s)-2\bar{G}(s)}{2s\left[3\bar{K}(s)+\bar{G}(s)\right]}. \tag{27a,b}$$

On the other hand, the Laplace transform of equation (24) for the dilatational response gives

$$\underbrace{\left(1+P_1 s+P_2 s^2+\cdots\right)}_{P(s)}\bar{\sigma}(s)-\frac{1}{s}\sum_{k=1}^{N}P_k\sum_{r=1}^{k}s^r\sigma^{(k-r)}(0)$$
$$=\underbrace{\left(Q_0+Q_1 s+Q_2 s^2+\cdots\right)}_{Q(s)}\bar{\varepsilon}(s)-\frac{1}{s}\sum_{k=1}^{N}Q_k\sum_{r=1}^{k}s^r\varepsilon^{(k-r)}(0) \tag{28}$$

where $\sigma^{(k-r)}(0)$ designates the $(k-r)$ order derivative of σ evaluated at $t=0$, with a similar definition for $\varepsilon^{(k-r)}(0)$. Algebraic operators $P(s)$ and $Q(s)$ are the coefficients of $\bar{\sigma}(s)$ and $\bar{\varepsilon}(s)$ for the assumed dilatational viscoelastic model.

By comparing equation (28) with equation (22a), it is seen that

$$3s\bar{K}=\frac{Q(s)}{P(s)} \tag{29}$$

and

$$\sum_{r=k}^{N}P_r\sigma^{(k-r)}(0)=\sum_{r=k}^{N}Q_r\varepsilon^{(k-r)}(0) \qquad k=1,2,3,\ldots,N. \tag{30}$$

Equation (30) indicates that the initial conditions upon stress and strain are not completely independent and relations such as equation (30) must be satisfied.

Employing a similar procedure with respect to the shear response, gives

$$2s\bar{G}=\frac{q(s)}{p(s)} \tag{31}$$

$$\sum_{r=k}^{N}p_r S^{(k-r)}(0)=\sum_{r=k}^{N}q_r\varepsilon^{(k-r)}(0) \qquad k=1,2,3,\ldots,N \tag{32}$$

where $p_r, q_r, p(s), q(s)$ are analogous coefficients in the assumed viscoelastic model in shear response. Finally, combining equations (26a,b), (27a,b), (29) and (31) leads to

$$s\bar{E}(s)=\frac{3Q(s)q(s)}{2p(s)Q(s)+q(s)P(s)} \qquad s\bar{\nu}(s)=\frac{p(s)Q(s)-q(s)P(s)}{2p(s)Q(s)+q(s)P(s)}. \tag{33a,b}$$

3.2. *Helices with strands of Kelvin, Maxwell and Zener materials*

3.2.1. Helix with strand of a Kelvin material. Using two different Kelvin models for dilatational and shear deformations of the strand material, that is,

$$\sigma=Q_0\varepsilon+Q_1\dot{\varepsilon} \qquad S=q_0\varepsilon+q_1\dot{\varepsilon} \tag{34}$$

respectively, one can easily determine P, Q, p, q as

$$P(s) = 1 \qquad Q(s) = Q_0 + Q_1 s \tag{35a,b}$$
$$p(s) = 1 \qquad q(s) = q_0 + q_1 s. \tag{36a,b}$$

Hence,

$$\begin{aligned} s\bar{E}(s) &= \frac{3\left(q_0 Q_0 + (q_0 Q_1 + Q_0 q_1)s + q_1 Q_1 s^2\right)}{2Q_0 + q_0 + (2Q_1 + q_1)s} \\ s\bar{\nu}(s) &= \frac{Q_0 - q_0 + (Q_1 - q_1)s}{2Q_0 + q_0 + (2Q_1 + q_1)s}. \end{aligned} \tag{37a,b}$$

Substituting equations (37a,b) into (4a,b), adopting $\sigma = F/A$ for Cauchy stress, $\mu = M/A$ for couple-stress and extensively rearranging the terms, one can find the Laplace transforms of the desired differential equations as

$$\left(D_0 + D_1 s + D_2 s^2\right)\bar{\sigma} = \left(E_0 + E_1 s + E_2 s^2 + E_3 s^3\right)\bar{\varepsilon} + \left(B_0 + B_1 s + B_2 s^2 + B_3 s^3\right)\bar{\tau} \tag{38}$$

$$\left(D'_0 + D'_1 s + D'_2 s^2\right)\bar{\mu} = \left(E'_0 + E'_1 s + E'_2 s^2 + E'_3 s^3\right)\bar{\varepsilon} + \left(B'_0 + B'_1 s + B'_2 s^2 + B'_3 s^3\right)\bar{\tau}. \tag{39}$$

Here all the constant coefficients, Ds, Es, Bs and $D's$, $E's$, $B's$, are functions of the geometry and the Kelvin model parameters. As before, the bar sign refers to quantities in the Laplace space.

To find the differential equations of the helix, it now suffices to compare equations (38) and (39) to equations (24) and (28) so as to easily obtain

$$\begin{aligned} D_0\sigma + D_1\sigma^{(1)} + D_2\sigma^{(2)} &= E_0\varepsilon + E_1\varepsilon^{(1)} + E_2\varepsilon^{(2)} + E_3\varepsilon^{(3)} + B_0\tau \\ &\quad + B_1\tau^{(1)} + B_2\tau^{(2)} + B_3\tau^{(3)} \end{aligned} \tag{40}$$

$$\begin{aligned} D'_0\mu + D'_1\mu^{(1)} + D'_2\mu^{(2)} &= E'_0\varepsilon + E'_1\varepsilon^{(1)} + E'_2\varepsilon^{(2)} + E'_3\varepsilon^{(3)} + B'_0\tau \\ &\quad + B'_1\tau^{(1)} + B'_2\tau^{(2)} + B'_3\tau^{(3)}. \end{aligned} \tag{41}$$

(40) and (41) are the differential equations of a viscoelastic helix with $\nu(t) \neq 0$ for the material at the micro level. Here, for convenience of notation, we employ $^{(n)}$ to denote the n-th order time derivative. In view of equation (30), in order for equation (40) to be the inverse Laplace transform of equation (38), the following relations must be satisfied:

$$\sum_{r=k}^{N=3} D_r\sigma^{(k-r)}(0) = \sum_{r=k}^{N=3}\left[E_r\varepsilon^{(k-r)}(0) + B_r\tau^{(k-r)}(0)\right] \quad k = 1, 2, 3. \tag{42}$$

Equation (42) imposes three independent constraints upon initial stresses and strains. Similar conditions apply to equation (41). In the special case when $\nu = 0$, equation (33b) leads to $p(s)Q(s) = q(s)P(s)$, and this simplifies equation (33a) to

$$s\bar{E}(s) = \frac{q(s)}{p(s)} = \frac{Q(s)}{P(s)}. \tag{43}$$

If we now use (43), instead of equations (33a,b), along with $\nu = 0$, in the correspondence principle, then we arrive at much simpler forms of the differential equations of viscoelastic helix:

$$
\begin{aligned}
D_0\sigma &= E_0\varepsilon + E_1\dot{\varepsilon} + B_0\tau + B_1\dot{\tau} \\
D'_0\mu &= E'_0\varepsilon + E'_1\dot{\varepsilon} + B'_0\tau + B'_1\dot{\tau}.
\end{aligned} \tag{44a,b}
$$

Except for the obvious coupling terms, (44a,b) are similar in type to the Kelvin differential equations. It is only in this special case of strand's Poisson ratio equal to zero, that we have a Kelvin-type helix.

3.2.2. Helix with strand of a Maxwell material. By using two Maxwell models for the dilatational and shear responses of the strand material at the micro level

$$\sigma + P_1\dot{\sigma} = Q_1\dot{\varepsilon} \qquad S + p_1\dot{S} = q_1\dot{\varepsilon} \tag{45a,b}$$

and taking similar steps as those for the Kelvin model, we determine the differential equations of this viscoelastic helix

$$
\begin{aligned}
D_2\sigma^{(2)} + D_3\sigma^{(3)} + D_4\sigma^{(4)} + D_5\sigma^{(5)} &= E_3\varepsilon^{(3)} + E_4\varepsilon^{(4)} + E_5\varepsilon^{(5)} \\
&\quad + B_3\tau^{(3)} + B_4\tau^{(4)} + B_5\tau^{(5)}
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
D'_2\mu^{(2)} + D'_3\mu^{(3)} + D'_4\mu^{(4)} + D'_5\mu^{(5)} &= E'_3\varepsilon^{(3)} + E'_4\varepsilon^{(4)} + E'_5\varepsilon^{(5)} \\
&\quad + B'_3\tau^{(3)} + B'_4\tau^{(4)} + B'_5\tau^{(5)}.
\end{aligned} \tag{47}
$$

Here $\sigma^{(5)}$ and $\sigma^{(4)}$ are the fifth and fourth time derivatives of stress, and similarly for strain. All the constant coefficients, Ds, Es, Bs and $D's$, $E's$, $B's$, are functions of the geometry and the Maxwell model parameters. Again the initial conditions are not completely independent. For instance, considering (46) the following five equations must be satisfied:

$$\sum_{r=k}^{N=5} D_r\sigma^{(k-r)}(0) = \sum_{r=k}^{N=5}\left[E_r\varepsilon^{(k-r)}(0) + B_r\tau^{(k-r)}(0)\right] \quad k = 1, 2, 3, 4, 5. \tag{48}$$

Clearly, equations (46) and (47) are more complex in type than the original Maxwell material of the strand. Again it may be shown that, only in the special case of strand's Poisson ratio equal to zero, does one obtain a Maxwell-type helix

$$D_0\sigma + D_1\sigma^{(1)} = E_1\varepsilon^{(1)} + B_1\tau^{(1)} \tag{49}$$

$$D'_0\mu + D'_1\mu^{(1)} = E_1\varepsilon^{(1)} + B'_1\tau^{(1)}. \tag{50}$$

3.2.3. Helix with strand of a Zener material. The Zener (or 3-Parameter) solid is the simplest acceptable viscoelastic model sufficiently close to real solids [8]. By using two such models for the material at the micro level

$$\sigma + P_1\dot{\sigma} = Q_0\varepsilon + Q_1\dot{\varepsilon} \qquad S + p_1\dot{S} = q_0\varepsilon + q_1\dot{\varepsilon} \tag{51a,b}$$

and following similar procedures as those used for Kelvin and Maxwell models, one can find the differential equations of a corresponding viscoelastic helix:

$$\begin{aligned} D_0\sigma + D_1\sigma^{(1)} + D_2\sigma^{(2)} + D_3\sigma^{(3)} + D_4\sigma^{(4)} + D_5\sigma^{(5)} \\ = E_0\varepsilon + E_1\varepsilon^{(1)} + E_2\varepsilon^{(2)} + E_3\varepsilon^{(3)} + E_4\varepsilon^{(4)} + E_5\varepsilon^{(5)} \\ + B_0\tau + B_1\tau^{(1)} + B_2\tau^{(2)} + B_3\tau^{(3)} + B_4\tau^{(4)} + B_5\tau^{(5)} \end{aligned} \quad (52)$$

$$\begin{aligned} D'_0\mu + D'_1\mu^{(1)} + D'_2\mu^{(2)} + D'_3\mu^{(3)} + D'_4\mu^{(4)} + D'_5\mu^{(5)} \\ = E'_0\varepsilon + E'_1\varepsilon^{(1)} + E'_2\varepsilon^{(2)} + E'_3\varepsilon^{(3)} + E'_4\varepsilon^{(4)} + E'_5\varepsilon^{(5)} \\ + B'_0\tau + B'_1\tau^{(1)} + B'_2\tau^{(2)} + B'_3\tau^{(3)} + B'_4\tau^{(4)} + B'_5\tau^{(5)}. \end{aligned} \quad (53)$$

Relations between initial conditions similar to (48) applied here as well. The solution to all the above differential equations for creep and relaxation tests can be found by using the Laplace transform technique.

Clearly, equations (52)–(53) are more complex in type than the original Zener material of the strand. Again, here it may be easily shown that, only in the special case of the strand's Poisson ratio equal to zero, does one obtain a Zener-type helix:

$$D_0\sigma + D_1\sigma^{(1)} = E_0\varepsilon + E_1\varepsilon^{(1)} + B_0\tau + B_1\tau^{(1)} \quad (54)$$

$$D'_0\mu + D'_1\mu^{(1)} = E'_0\varepsilon + E'_1\varepsilon^{(1)} + B'_0\tau + B'_1\tau^{(1)}. \quad (55)$$

3.3. *Relaxation and creep tests*

Thus far, we have described the viscoelastic helix by its differential equations. To study the behaviour of such a helix, a standard test consisting of a creep and a relaxation test has to be used, e.g. [8]. In the relaxation test constant strains are applied and the time-dependent stresses are derived, and in the creep test constant stresses are applied and time-dependent strains are derived.

3.3.1. Relaxation test. In the relaxation test both ε and τ are assumed to be constant. Therefore

$$\dot{\varepsilon} = \ddot{\varepsilon} = \cdots = \dot{\tau} = \ddot{\tau} = \cdots = 0. \quad (56)$$

The differential equations of a viscoelastic helix using two Kelvin models (40) and (41) simplify to

$$D_0\sigma + D_1\dot{\sigma} + D_2\ddot{\sigma} = E_0\varepsilon + B_0\tau \qquad D'_0\mu + D'_1\dot{\mu} + D'_2\ddot{\mu} = E'_0\varepsilon + B'_0\tau. \quad (57a,b)$$

Equations (57a,b) are two independent linear second-order differential equations with constant coefficients. The time-dependent stresses for the Kelvin model in terms of hyperbolic functions are

$$\sigma_{\text{Kelvin}} = \frac{(\varepsilon E_0 + \tau B_0)}{D_0}\left[1 + e^{(-D_1/(2D_2)t)}\left(-\cosh\left(\frac{t\sqrt{D_1^2 - 4D_0D_2}}{2D_2}\right) - \frac{D_1 \sinh\left(t\sqrt{D_1^2 - 4D_0D_2}/(2D_2)\right)}{\sqrt{D_1^2 - 4D_0D_2}}\right)\right] \tag{58}$$

$$S_{\text{Kelvin}} = \frac{\left(\varepsilon E'_0 + \tau B'_0\right)}{D'_0}\left[1 + e^{(-D'_1/(2D'_2)t)}\left(-\cosh\left(\frac{t\sqrt{D_1'^2 - 4D'_0D'_2}}{2D'_2}\right) - \frac{D'_1 \sinh\left(t\sqrt{D_1'^2 - 4D'_0D'_2}/(2D_2)\right)}{\sqrt{D_1'^2 - 4D'_0D'_2}}\right)\right] \tag{59}$$

and the time-dependent stresses for Zener models are

$$\sigma_{\text{Zener}} = \frac{-(\varepsilon E_0 + \tau B_0)}{D_0}\left[\left(\sum_{i=1}^{5}\frac{\left(D_5\lambda_i^4 + D_4\lambda_i^3 + D_3\lambda_i^2 + D_2\lambda_i + D_1\right)}{5D_5^4\lambda_i^4 + 4D_4\lambda_i^3 + 3D_3\lambda_i^2 + 2D_2\lambda_i + D_1}e^{\lambda_i t}\right) - 1\right] \tag{60}$$

$$S_{\text{Zener}} = \frac{-\left(\varepsilon E'_0 + \tau B'_0\right)}{D'_0}\left[\left(\sum_{i=1}^{5}\frac{\left(D'_5\lambda_i^4 + D'_4\lambda_i^3 + D'_3\lambda_i^2 + D'_2\lambda_i + D'_1\right)}{5D_5'^4\lambda_i^4 + 4D'_4\lambda_i^3 + 3D'_3\lambda_i^2 + 2D'_2\lambda_i + D'_1}e^{\lambda_i t}\right) - 1\right]. \tag{61}$$

In the equation (60)

$$\lambda_i = \text{Root of }\left(D_5z^5 + D_4z^4 + D_3z^3 + D_2z^2 + D_1z + D_0\right) \tag{62}$$

while in (61)

$$\lambda_i = \text{Root of }\left(D'_5z^5 + D'_4z^4 + D'_3z^3 + D'_2z^2 + D'_1z + D'_0\right). \tag{63}$$

In general, in the relaxation test of the viscoelastic helix — given that both strains ε and τ are constant and stresses are uncoupled — each of the differential equations can be solved independently to find the time-dependent stresses. In the creep test, however, coupled differential equations need to be solved to find the time-dependent strains.

3.3.2. Creep test. Creep is a situation where, with the passage of time, stresses remain constant and deformations continue to grow. This time-dependent phenomenon involves a time-dependent PR. Now, such PR, or $\nu(t)$, has been defined in several forms in the literature [5], but the one that is amenable to the correspondence principle is defined as minus the ratio of the time-dependent lateral strain to the constant axial strain, under stress relaxation conditions [5, 7]. In the creep test, both stresses σ and μ are assumed to be constant and since strains are varying, one cannot use equation (26b). In order to be able to use the correspondence principle for the creep test and be consistent with the definition of v, one has to replace v by equation (25b) in the elastic solution and work with K and G to find the viscoelastic response. Interestingly, this leads to the same differential equations as we already derived for Kelvin, Maxwell and Zener models. For example, for the Kelvin model at a strand's level, by recourse to the aforementioned method, we end up deriving the same equations as in (40) and (41).

For the creep test, recalling the assumption of the standard test [8, 9], we have

$$\dot{\sigma} = \ddot{\sigma} = \cdots = \dot{\mu} = \ddot{\mu} = \cdots = 0 \tag{64}$$

and, upon taking the Laplace transform of equations (40) and (41), we find

$$D_0 \frac{\sigma}{s} = \left(E_3 s^3 + E_2 s^2 + E_1 s + E_0\right)\bar{\varepsilon} + \left(B_3 s^3 + B_2 s^2 + B_1 s + B_0\right)\bar{\tau} \tag{65}$$

$$D_0' \frac{\mu}{s} = \left(E_3' s^3 + E_2' s^2 + E_1' s + E_0'\right)\bar{\varepsilon} + \left(B_3' s^3 + B_2' s^2 + B_1' s + B_0'\right)\bar{\tau}. \tag{66}$$

To find the solution, unlike the relaxation case, we have to solve a system of two coupled differential equations to find the two unknowns $\bar{\varepsilon}$, $\bar{\tau}$. Thus,

$$\begin{aligned}\bar{\varepsilon} = {} & D_0\left(B_3' s^4 + B_2' s^3 + B_1' s^2 + B_0' s\right)\sigma \\ & - D_0'\left(B_3 s^4 + B_2 s^3 + B_1 s^2 + B_0 s\right)\mu \Big/ \Big[\left(E_3 s^4 + E_2 s^3 + E_1 s^2 + E_0 s\right) \\ & \times \left(B_3' s^4 + B_2' s^3 + B_1' s^2 + B_0' s\right) - \left(E_3' s^4 + E_2' s^3 + E_1' s^2 + E_0' s\right) \\ & \times \left(B_3 s^4 + B_2 s^3 + B_1 s^2 + B_0 s\right)\Big]\end{aligned} \tag{67}$$

$$\begin{aligned}\bar{\tau} = {} & D_0'\left(E_3 s^4 + E_2 s^3 + E_1 s^2 + E_0 s\right)\mu - D_0\left(E_3' s^4 + E_2' s^3 + E_1' s^2 + E_0' s\right)\sigma \\ & \Big/ \Big[\left(E_3 s^4 + E_2 s^3 + E_1 s^2 + E_0 s\right)\left(B_3' s^4 + B_2' s^3 + B_1' s^2 + B_0' s\right) \\ & - \left(E_3' s^4 + E_2' s^3 + E_1' s^2 + E_0' s\right)\left(B_3 s^4 + B_2 s^3 + B_1 s^2 + B_0 s\right)\Big].\end{aligned} \tag{68}$$

By performing the inverse Laplace transform on equations (67) and (68), the time-dependent forms of ε and τ are found to be

$$\varepsilon = \sum_{i=1}^{8} \frac{J(\lambda_i)}{I'(\lambda_i)} e^{\lambda_i t} \qquad \tau = \sum_{i=1}^{8} \frac{H(\lambda_i)}{I'(\lambda_i)} e^{\lambda_i t}. \tag{69}$$

Here I is the denominator of (67) and (68) — indeed, a polynomial function of s — while λ_is are the roots of this polynomial; $I'(\lambda_i)$ is the derivative of I with respect to s calculated at λ_i; J and H are numerators of equations (67) and (68), respectively, which are polynomial functions of s calculated at λ_i. By using the same procedure, one can find ε and τ for any other models, like the Maxwell model or the

Zener model. More extensive details on those and all other aspects of the present study can be found in [10].

4. Results and conclusions

In the case of an elastic helix:

- By reworking the numerical-type derivation of Costello [2] in a more concise form, all the constitutive helix coefficients have been derived explicitly.
- Since the ratio C_3/C_2 comes out never equal to unity unless $\alpha = \pi/2$, further work on its improved analytical derivation is needed. This should allow a better assessment of the validity or influence of the various hypotheses involved in the mechanics of a single helically shaped strand (or a bundle of strands). The key assumptions that cause this discrepancy are: the product of higher-order terms resulting from the strain of a single helical strand is neglected; the changes in the curvature and twist per unit length are linearized; small $\Delta\alpha = \alpha_2 - \alpha_1$ has been assumed and, based on this, some trigonometric functions are simplified accordingly.
- C_2 and C_3 do not depend on the centre strand and the only contribution of the centre strand is in the first terms of C_1 and C_4.
- When α approaches $\pi/2$, C_1 and C_4 approach values proportional to E and G, respectively, while C_2 and C_3 approach zero.
- The maximum value of C_1 occurs at $\alpha = \pi/2$ while the maximum values of C_2 and C_3 occur approximately when $\alpha = 1$ rad. Also, the maximum value of C_4 occurs when α is about 0.6–0.7 rad.
- The ratio C_3/C_2 gets closer to unity as α approaches $\pi/2$. This observation turns out to be a general trend for any helical strand.
- The maximum difference between strain energies involved in the C_2 and C_3 terms is about 20%.

In the case of viscoelastic helices:

- The differential equations of a linear viscoelastic helix considering the time-dependent PR have been derived, along with explicit forms of all the coefficients, for three basic types of linear differential models (Kelvin, Maxwell, Zener) for the material at the micro level.
- The linear viscoelastic responses of a helix (either a bundle or a single strand) are generally different in type (and more complex) from those of the material at the micro level.
- When $v(t) = 0$, the type of viscoelastic response of the helix is qualitatively the same as the viscoelastic response of the assumed model at the micro level.
- Although we did not work with an arbitrary-order differential equation (24), our study clearly indicates that direct viscoelastic generalizations of effective constitutive equations of helices, not based on systematic analyses such as those presented here, are going to be invalid. This is due to the fact that, for a given complexity of the material model, higher-order derivatives are showing up in the differential equation governing the helix.

- The foregoing observation relating to uniaxial helices also provides guidance for admissible *vis-à-vis* inadmissible models of three-dimensional (3-D) chiral (i.e. helically structured) materials. It is well known that the constitutive relation of a linear elastic chiral material in 3-D involves two coupled equations [11]

$$\sigma_{ij} = C^{(1)}_{ijkl}\varepsilon_{kl} + C^{(2)}_{ijkl}\kappa_{kl} \tag{70}$$

$$\mu_{ij} = C^{(3)}_{ijkl}\varepsilon_{kl} + C^{(4)}_{ijkl}\kappa_{kl} \tag{71}$$

where μ_{ij} is the couple-stress tensor, while κ_{kl} is the torsion–curvature tensor. In view of the consequences obtained in section 3, one cannot arbitrarily postulate viscoelastic generalizations of (70) and (71) in terms of differential tensorial equations. Another case study where a uniaxial helix model based on equations (4a,b) has provided guidance on thermomechanics of 3-D chiral materials has recently been presented in [12]. Other studies employing the same Ansatz were focused on vibration responses of a single-phase helix [13] and homogenization of microstructured elastic and thermoelastic helices [14, 15].

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